Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 20, No. 1, 101-112

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On p-adic L-functions attached to motives over Q II

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Abstract. We propose a definition of the p-adic L-function of a motive M over $\mathbb{Q}$, assuming M admits at least one critical point, and p is ordinary for M. This corrects by a power of $i=\sqrt{-1}$ an earlier definition of B. Perrin-Riou and the author.

Introduction

Let M be a motive over $\mathbb{Q}$ which admits at least one critical point $s \in \mathbb{Z}$ in the sense of Deligne [3], and let p be a prime number which is ordinary for M. In a previous paper [1], Bernadette Perrin-Riou and I conjectured the existence of certain p-adic measures, which provide the p-adic analogue of the complex L-series of M. In a letter to us, Deligne has pointed out that there is a more elegant and succinct way of expressing our conjecture, by using the local ϵ-factors of M. Also, in some cases, it is clear form his remark that the conjecture of [1] should be modified by a suitable power of $i = \sqrt{-1}$, which depends on the ϵ-factor at ∞ of M. Thus the aim of the present note is to give a new (and hopefully now correct) formulation of the conjecture of [1], based on Deligne's observation. I also give some refinements and re-interpretations of the preliminary arguments of [1], involving the crucial modifications of the Euler factors at ∞ and p of the complex L-series of M. In addition, I have changed normalizations so that the given critical point in this note is $s = 0$ (as in [3]), rather than $s = 1$ (as in [1]). I am now fully convinced that this normalization is the most natural one from all points of view, including the connexion with Iwasawa modules (which we do not discuss here). For simplicity, I have tried wherever possible to folow the notation of [3] in the present note. Finally, I would like to thank P. Deligne for his very helpful criticisms of [1].

1. Modification of the Euler factor at ∞

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For each prime v of $\mathbb{Q}$, $\mathbb{Q}_v$ will

Received 10 April 1989.

denote the completion at v, $\overline{\mathbb{Q}}_v$ and algebraic closure of $\mathbb{Q}_v$, and $G(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ the associated Galois group.

As above, let M be a motive over $\mathbb{Q}$, which is homogeneous of weight $\omega(M)$. We follow the notation of [3]. Thus F_∞ will denote the involution of the Betti realisation $H_B(M)$ which is induced by complex conjugation. Let $d^+(M)$ be the $\mathbb{Q}$-dimension of the subspace of $H_B(M)$ fixed by F_∞. We write $\check{M}$ for the dual motive of M, and, for each $n \in \mathbb{Z}$, $M(n)$ will denote the n-fold twist of M by the Tate motive $\mathbb{Q}(n)$.

We only briefly recall the theory of the complex L-series attached to M. For each prime v of $\mathbb{Q}$, let $L_v(M,s)$ denote the classical Euler factor attached to v (including $v = \infty$). The global L-series is then the Euler product

$$\bigwedge(M,s) = \prod_v L_v(M,s),$$

which converges in the half plane $R(s) > 1 + \omega(M)/2$. The principal conjecture of the complex theory (which we shall tacitly assume) asserts that $\bigwedge(M,s)$ has a meromorphic continuation over the whole complex plane to a function of order ≤ 1, and satisfies the functional equation

$$\bigwedge(M,s) = \varepsilon(M,s)\bigwedge(\check{M}(1),-s), \tag{1}$$

where $\varepsilon(M,s)$ is Deligne's global ε-factor, normalized as in [2]. Recall that a point $s = n$ in $\mathbb{Z}$ is said to be *critical* for M if both the Euler factors at infinity $L_\infty(M,s)$ and $L_\infty(\check{M}(1),-s)$ are holomorphic at $s = n$. Throughout this paper, we assume the

Hypothesis on M. *The point $s = 0$ is critical for M.*

Note that this is a different normalization from [1], where $s = 1$ was taken to be the fixed critical point. Standard conjectures about the possible poles of $\bigwedge(M,s)$ (which we shall assume) then imply that $\bigwedge(M,s)$ is also holomorphic at $s = 0$. Following [3], we shall write

$$\bigwedge(M) = \bigwedge(M,0), \quad L(M) = L(M,0), \quad \varepsilon(M) = \varepsilon(M,0). \tag{2}$$

Note that, because of different normalizations, the above $\varepsilon(M)$ is not the same as that in [1].

One of the delicate points of the complex theory – which we shall see also turns out to be basic for the non-archimedian theory – is that the global factor $\varepsilon(M)$ can be written as a product of local ε-factors (see [2], and also [4]). Let $\mathbb{A}$ denote the adèle group of $\mathbb{Q}$. Fix, once and for all, the Haar measure $dx = \prod dx_v$ on $\mathbb{A}$, where dx_∞ is the usual measure on $\mathbb{R}$, and, for each finite prime q, dx_q is the Haar measure on $\mathbb{Q}_q$ which gives $\mathbb{Z}_q$ volume 1. For simplicity, we suppress all reference to this fixed measure in the subsequent notation. We must also fix

an (additive) character of $\mathbb{A}/\mathbb{Q}$, and there are two natural choices. Let $\psi^{(i)}$ denote the character of $\mathbb{A}/\mathbb{Q}$ with components $\psi^{(i)}_\infty(x) = \exp(2\pi i x)$, and, for each finite q, $\psi^{(i)}_q(x) = \exp(-2\pi i x)$, where we have identified $\mathbb{Q}_q/\mathbb{Z}_q$ with the q-primary part of $\mathbb{Q}/\mathbb{Z}$. The second natural choice is $\psi^{(-i)}(x) = \psi^{(i)}(-x)$. For the rest of this article, ρ will denote one of $\pm i$. We then have

$\rho = \pm i$ reflects choice of ψ (?)

$$\varepsilon(M) = \prod_v \varepsilon_v(M, \psi^{(\rho)}),$$

where $\varepsilon_v(M,\psi^{(\rho)})$ denotes Deligne's local ε_v-factor (with the measure dx_v dropped from the notation), and the product is taken over all primes v of $\mathbb{Q}$. Note that we have

$$\varepsilon_v(M,\psi^{(\rho)})\varepsilon_v(\check{M}(1),\psi^{(-\rho)}) = 1. \tag{3}$$

We also recall another operation on motives, which is of greater importance for the study of non-archimedian L-functions than for the complex L-functions. Let χ be a Dirichlet character of $\mathbb{Q}$, and write $C(\chi)$ for its conductor. Let $\mu_{C(\chi)}$ denote the group of $C(\chi)$-th roots of unity in $\overline{\mathbb{Q}}$, and let $\mathcal{G}_\chi$ denote the Galois group of the field generated over $\mathbb{Q}$ by $\mu_{C(\chi)}$. We can identify $\mathcal{G}_\chi$ with $(\mathbb{Z}/C(\chi)\mathbb{Z})^*$ via the action of $\mathcal{G}_\chi$ on $\mu_{C(\chi)}$, and thus we can identify χ with a character of $\mathcal{G}_\chi$ and so also of the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. This defines the ℓ-adic realisations of χ (these are 1-dimensional vector spaces over the completions at the primes above ℓ of any finite extension of $\mathbb{Q}$ containing the values of χ). One can also define Betti and de Rham realisations of χ (see §6 of [3]) and thus attach a motive $[\chi]$ to χ. We then define $M(\chi)$ to be the motive over $\mathbb{Q}$ whose realisations are the tensor products of the realisations of M with the realisations of $[\chi]$.

We now define the modified Euler factor at ∞, which we denote by $\mathcal{L}^{(\rho)}_\infty(M)$, and which, as indicated, will depend on the choice of $\rho = \pm i$. Recall that the usual Euler factor at ∞ depends only on the Hodge decomposition of $H_B(M)\otimes\mathbb{C}$, together with the $\mathbb{C}$-linear involution F_∞ of this space. It is given by

$$L_\infty(M) = \prod_U L_\infty(U),$$

where U runs over the summands of $H_B(M)\otimes\mathbb{C}$ of the form either $U = H^{(j,k)}(M) \oplus H^{(k,j)}(M)$ with $j < k$, or $U = H^{(j,j)}(M)$ (the exact definition of $L_\infty(U)$ is recalled, as needed, in the proof of the next lemma). The modified Euler factor at ∞ is then defined by

$$\mathcal{L}^{(\rho)}_\infty(M) = \prod_U \mathcal{L}^{(\rho)}_\infty(U), \tag{4}$$

where, putting $H^{(j,k)} = H^{(j,k)}(M)$ and $h(j,k) = \mathbb{C}$-dimension of $H^{(j,k)}$, we have

(a) If $U = H^{(j,k)} \oplus H^{(k,j)}$ with $j < k$, then $\mathcal{L}^{(\rho)}_\infty(U) = \rho^{jh(j,k)}L_\infty(U)$

(b) If $U = H^{(j,j)}$ with $j \geq 0$, then $\mathcal{L}_{\infty}^{(\rho)}(U) = 1$;

(c) If $U = H^{(j,j)}$ with $j < 0$, then

$$\mathcal{L}_{\infty}^{(\rho)}(U) = L_{\infty}(U)/(\varepsilon_{\infty}(U, \psi^{(\rho)})L_{\infty}(\check{U}(1)))$$

The explicit value of $\varepsilon_{\infty}(U, \psi^{(\rho)})$ is given in the table of p. 329 of [3]. This table, together with (3), shows that in case (a), we have

$$\varepsilon_{\infty}(U, \psi^{(\rho)}) = \rho^{(k-j+1)h(j,k)}$$

Note also that case (b) holds for U if and only if case (c) holds for $\check{U}(1)$. In view of these remarks, it is clear that the modified L-function

$$\Lambda_{(\infty)}^{(\rho)}(M) = \mathcal{L}_{\infty}^{(\rho)}(M)L(M) \tag{5}$$

satisfies the functional equation

$$\Lambda_{(\infty)}^{(\rho)}(M) = \prod_{v \neq \infty} \varepsilon_v(M, \psi^{(\rho)}).\Lambda_{(\infty)}^{(-\rho)}(\check{M}(1)). \tag{6}$$

Up to a change of normalization and a power of i, $\mathcal{L}_{\infty}^{(\rho)}(M)$ is the same as the modified Euler factor introduced on p. 37 of [1], and we owe to Deligne the suggestion to also transfer the ε-factors as given in (a), (b) and (c) above. That his suggestion works beautifully is shown by the validity of the following strengthened form of Lemma 2.4 of [1]. If x, y are complex numbers, we write $x \sim y$ if there exists $a \neq 0$ in $\mathbb{Q}$ such that $x = ay$.

Lemma 1. *Let χ be a Dirichlet character, and $n \in \mathbb{Z}$ be such that $\chi(-1) = (-1)^n$ and $M(n)(\chi)$ is also critical at $s = 0$. Then*

$$\mathcal{L}_{\infty}^{(\rho)}(M(n)(\chi)) \sim (2\pi i)^{-nd^+(M)}\mathcal{L}_{\infty}^{(\rho)}(M). \tag{7}$$

twisting behaves well with rationality (twist to get values of p-adic L-fn) (?)

Proof. Note that the weight of $M(n)(\chi)$ is equal to $\omega(M) - 2n$. Also $d^+(M(n)(\chi)) = d^+(M)$ because $\chi(-1) = (-1)^n$. The argument breaks up into three main cases, according to the three possible choices for U given above. Put $d^+(U) = h(j,k)$ in case (a), $d^+(U) = 0$ in case (b), and $d^+(U) = h(j,j)$ in case (c). We shall prove that $d^+(M) = \sum_U (d^+(U)$, and that

$$\mathcal{L}_{\infty}^{(\rho)}(U(n)(\chi)) \sim (2\pi i)^{-nd^+(U)}\mathcal{L}_{\infty}^{(\rho)}(U), \tag{8}$$

which plainly establishes (7). Put $W = U(n)(\chi)$.

Case (a). $U = H^{(j,k)} \oplus H^{(k,j)}$ with $j < k$. Then $W = H^{(j-n,k-n)} \oplus H^{(k-n,j-n)}$. By definition, we have

$$L_{\infty}(U) = \Gamma_{\mathbb{C}}(-j)^{h(j,k)}, \quad L_{\infty}(W) = \Gamma_{\mathbb{C}}(n-j)^{h(j,k)}.$$

Recalling that $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) \sim (2\pi)^{-s}$ for $s > 0$ in $\mathbb{Z}$, it follows from (a) that

$$\mathcal{L}_{\infty}^{(\rho)}(U) \sim (2\pi i)^{jh(j,k)}, \quad \mathcal{L}_{\infty}^{(\rho)}(W) \sim (2\pi i)^{(j-n)h(j,k)},$$

whence (8) is clear in this case.

Case (b). $U = H^{(j,j)}$ with $j \geq 0$. We first show that F_{∞} always acts on U by -1, so that U contributes nothing to $d^+(M)$. For brevity, write $h = h(j,j)$. If F_{∞} acts on U as $(-1)^j$, then $L_{\infty}(U) = \Gamma_{\mathbb{R}}(-j)^h$, whence j is odd because $j \geq 0$. If F_{∞} acts on U as $(-1)^{j+1}$, then $L_{\infty}(U) = \Gamma_{\mathbb{R}}(1-j)^h$, whence j is even since $j \geq 0$. Thus F_{∞} always acts on U by -1. To complete the proof of (8), we must show that $j - n \geq 0$, since then

$$\mathcal{L}_{\infty}^{(\rho)}(U) = \mathcal{L}_{\infty}^{(\rho)}(W) = 1.$$

Case (b1). Assume j is odd. If n is even, $\chi(-1) = 1$, and so F_{∞} acts on W by $(-1)^{j-n}$, whence $L_{\infty}(\check{W}(1)) = \Gamma_{\mathbb{R}}(j-n+1)^h$. But $j-n+1$ is even, and so we must have $j - n \geq 0$. If n is odd, $\chi(-1) = -1$, and F_{∞} acts on W by $(-1)^{j-n+1}$, whence $L_{\infty}(\check{W}(1)) = \Gamma_{\mathbb{R}}(j-n+2)^h$. But $j-n+2$ is even, and so we must have $j \geq n$, as required.

Case (b2). Assume j is even. If n is even, $\chi(-1) = 1$, and F_{∞} acts on W by $(-1)^{j-n+1}$, whence $L_{\infty}(\check{W}(1)) = \Gamma_{\mathbb{R}}(j-n+2)^h$. But $j-n+2$ is even, whence $j \geq n$. If n is odd, $\chi(-1) = -1$, and F_{∞} acts on W by $(-1)^{j-n}$, whence $L_{\infty}(\check{W}(1)) = \Gamma_{\mathbb{R}}(j-n+1)^h$. But $j-n+1$ is even, and so again $j \geq n$.

Case (c). $U = H^{(j,j)}$ with $j < 0$. We first show that F_{∞} always acts on U by $+1$, so that U contributes $h = h(j,j)$ to $d^+(M)$. If F_{∞} acts on U as $(-1)^j$, then $L_{\infty}(\check{U}(1)) = \Gamma_{\mathbb{R}}(j+1)^h$, whence j is even since $j < 0$. If F_{∞} acts on U as $(-1)^{j+1}$, then $L_{\infty}(\check{U}(1)) = \Gamma_{\mathbb{R}}(j+2)^h$, whence j is odd because $j < 0$. Thus F_{∞} always acts on U by $+1$.

We next recall that, for $s \in \mathbb{Z}$, we have $\Gamma_{\mathbb{R}}(s) \sim (2\pi)^{(1-s)/2}$ for s odd, $\Gamma_{\mathbb{R}}(s) \sim (2\pi)^{-s/2}$ for s even and > 0.

Case (c1). Assume j is even. We shall show that

$$\mathcal{L}_{\infty}^{(\rho)}(U) \sim (2\pi)^{jh}, \quad \mathcal{L}_{\infty}^{(\rho)}(W) \sim (2\pi)^{(j-n)h}i^{nh}, \tag{9}$$

which plainly implies (8). Indeed,

$$L_{\infty}(U) = \Gamma_{\mathbb{R}}(-j)^h, \quad L_{\infty}(\check{U}(1)) = \Gamma_{\mathbb{R}}(j+1)^h, \quad \varepsilon_{\infty}(U, \psi^{(\rho)}) = 1.$$

Hence

$$L_{\infty}(U) \sim (2\pi)^{jh/2}, \quad L_{\infty}(\check{U}(1)) \sim (2\pi)^{-jh/2},$$

and the first assertion of (9) follows immediately. Suppose now that n is even, so that $\chi(-1) = 1$. Hence F_{∞} acts on W by $(-1)^{j-n}$, and so

$$L_{\infty}(W) = \Gamma_{\mathbb{R}}(n-j)^h, \quad L_{\infty}(\check{W}(1)) = \Gamma_{\mathbb{R}}(j-n+1)^h, \quad \varepsilon_{\infty}(W, \psi^{(\rho)}) = 1.$$

Now $j - n$ is even, and so $j - n < 0$. We obtain

$$L_{\infty}(W) \sim (2\pi)^{(j-n)h/2}, \quad L_{\infty}(\check{W}(1)) \sim (2\pi)^{(n-j)h/2},$$

and the second assertion of (9) follows in this case. Suppose next that n is odd,

so that $\chi(-1)=-1$. Hence F_∞ acts on W by $(-1)^{j-n+1}$, and so
$L_\infty(W)=\Gamma_{\mathbb{R}}(n-j+1)^h$, $L_\infty(\check{W}(1))=\Gamma_{\mathbb{R}}(j-n+2)^h$, $\varepsilon_\infty(W,\psi^{(\rho)})=\rho^h$.
Now $j-n-1$ is even, and so $j-n<0$. We obtain

$$L_\infty(W)\sim(2\pi)^{(j-n-1)h/2},\quad L_\infty(\check{W}(1))\sim(2\pi)^{(n-j-1)h/2},$$

and again the second assertion of (9) is plain.

Case (c2). Assume j is odd. We shall show that

$$L_\infty^{(\rho)}(U)\sim(2\pi)^{jh}i^h,\quad L_\infty^{(\rho)}(W)\sim(2\pi)^{(j-n)h}i^{(n-1)h}, \tag{10}$$

which plainly implies (8). Indeed

$$L_\infty(U)=\Gamma_{\mathbb{R}}(1-j)^h,\quad L_\infty(\check{U}(1))=\Gamma_{\mathbb{R}}(j+2)^h,\quad \varepsilon_\infty(U,\psi^{(\rho)})=\rho^h.$$

Hence

$$L_\infty(U)\sim(2\pi)^{(j-1)h/2},\quad L_\infty(\check{U}(1))\sim(2\pi)^{-(j+1)h/2},$$

and the first assertion of (10) is clear. Suppose now that n is even, so that $\chi(-1)=1$. Hence F_∞ acts on W by $(-1)^{j-n+1}$, and so
$L_\infty(W)=\Gamma_{\mathbb{R}}(n+1-j)^h$, $L_\infty(\check{W}(1))=\Gamma_{\mathbb{R}}(j-n+2)^h$, $\varepsilon_\infty(W,\psi^{(\rho)})=\rho^h$.
Now $j-n-1$ is even, whence $j-n<0$. We obtain

$$L_\infty(W)\sim(2\pi)^{(j-n-1)h/2},\quad L_\infty(\check{W}(1))\sim(2\pi)^{(n-j-1)h/2}.$$

Since n is even, the second assertion of (10) is now clear in this case. Suppose finally that n is odd, so that $\chi(-1)=-1$. Hence F_∞ acts on W by $(-1)^{j-n}$, and so

$$L_\infty(W)=\Gamma_{\mathbb{R}}(n-j)^h,\quad L_\infty(\check{W}(1))=\Gamma_{\mathbb{R}}(j-n+1)^h,\quad \varepsilon_\infty(W,\psi^{(\rho)})=1$$

Now $j-n$ is even, and thus $j-n<0$. We obtain

$$L_\infty(W)\sim(2\pi)^{(j-n)h/2},\quad L_\infty(\check{W}(1))\sim(2\pi)^{(n-j)h/2}.$$

As n is odd, the second assertion of (10) now follows in this case. This completes the proof of Lemma 1. ☐

Note that the proof of Lemma 1 also shows that

$$d^+(M)=\sum_{j<0}h(j,k). \tag{11}$$

Let us also define

$$\tau(M)=\sum_{j<0}jh(j,k). \tag{12}$$

We can now give an equivalent form of Deligne's period conjecture in [3], which is better suited for questions of p-adic interpolation. Let $C^+(M)$ be the period defined on p. 320 of [3]. Recall also that $C^+(M)$ is only determined up to multiplication by a non-zero element of $\mathbb{Q}$. Having made a choice of $C^+(M)$,

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we define

$$\Omega^{(\rho)}(M)=C^+(M)(2\pi\rho)^{\tau(M)}.$$

The arguments in the proof of Lemma 1 show immediately that

$$\Omega^{(\rho)}(M)\sim C^+(M)\mathcal{L}_\infty^{(\rho)}(M).$$

Again using Lemma 1, we therefore obtain the following equivalent form of the period conjecture of [3].

Period Conjecture. *Let χ be a Dirichlet character and $n\in\mathbb{Z}$ be such that $\chi(-1)=(-1)^n$ and $M(n)(\chi)$ is critical at $s=0$. Then*

$$\Lambda_{(\infty)}^{(\rho)}(M(n)(\chi)).\Omega^{(\rho)}(M)^{-1}\in\overline{\mathbb{Q}}.$$

The following Lemma is implicit in the proof of Lemma 1, but we record it explicitly as we shall apply it several times.

Lemma 2. *Assume that $\chi(-1)=(-1)^n$ and $M(n)(\chi)$ is critical at $s=0$. If $h(j,k)\neq 0$, then $j<0$ if and only if $j<n$.*

Proof. Assume $j<0$. The fact that M is critical at $s=0$ implies that $j\le k$, and then it is shown in the proof of Lemma 1 that $j<n$. If we assume $j<n$, we apply the previous reasoning with M replaced by $N=M(n)(\chi)$ and $N(-n)(\chi^{-1})$. ☐

We now briefly mention two functorial properties of our periods $\Omega^{(\rho)}(M)$. With χ and n as in the period conjecture, we have

$$\frac{\Lambda_{(\infty)}^{(\rho)}(M(n)(\chi))}{\Omega^{(\rho)}(M)}=(-1)^{nd^+(M)}\frac{\Lambda_{(\infty)}^{(-\rho)}(M(n)(\chi))}{\Omega^{(-\rho)}(M)} \tag{13}$$

Obviously we have the identity.

$$\frac{\Omega^{(\rho)}(M)}{\Omega^{(-\rho)}(M)}=(-1)^{\tau(M)}$$

On the other hand, the formulae in the proof of Lemma 1 show that

$$\frac{\mathcal{L}_\infty^{(\rho)}(M)}{\mathcal{L}_\infty^{(-\rho)}(M)}=(-1)^{\tau(M)}.$$

We obtain (13) by applying this last identity to $M(n)(\chi)$, and noting that Lemma 2 shows that $\tau(M(n)(\chi))=\tau(M)-nd^+(M)$. The second functorality concerns the functional equation. In view of (6), we would expect

$$\Omega^{(-\rho)}(\check{M}(1))\sim\Omega^{(\rho)}(M)/(\prod_{v\neq\infty}\varepsilon_v(M,\psi^{(\rho)})).$$

This can indeed be verified using the arguments of §5 of [3] (but one must assume the additional Conjecture 6.6).

2. Modification of the Euler factor at p

Let p be a prime number such that M has good reduction at p, i.e., for each prime $\ell \neq p$, the inertial subgroup I_p of $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ operates trivially on the ℓ-adic realisation of M, which we denote by $H_\ell(M)$. We shall also consider the twist of M by an arbitrary Dirichlet character χ ($M(\chi)$ will not, in general, have good reduction at p). In [1], we introduced a modification of the Euler factor at p of $M(\chi)$. We now explain how our earlier modification can be viewed as parallel to that given for the Euler factor at ∞ in §1. The reader will also notice that, unlike the case at ∞, the modification of the p-Euler factor does not depend on our hypothesis that M is critical at $s = 0$; indeed, even when M is critical at $s = 0$, $L_p(M, s)$ may have a pole at $s = 0$.

Fix, once and for all, an embedding

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p. \tag{14}$$

Recall that we assume that, for all $\ell \neq p$, $\det(1 - Frob_p^{-1}X|H_\ell(M))$ has coefficients in $\mathbb{Q}$ independent of ℓ. Write $P(M)$ for the set of inverse roots α of this polynomial in $\overline{\mathbb{Q}}$, always taken with multiplicity. By virtue of the embedding (14), we can talk of the p-adic order $\mathrm{ord}_p(\alpha)$ of each $\alpha \in P(M)$.

Let ℓ be a fixed prime $\neq p$. Pick an embedding of Q_ℓ into the complex field $\mathbb{C}$. Let $J_\ell(M)$ denote the semi-simplification of $H_\ell(M) \otimes \mathbb{C}$ as a representation of $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Thus

$$J_\ell(M) = \oplus_\alpha U_\alpha,$$

where α runs over $P(M)$, and U_α is a 1-dimensional complex representation of $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (i.e. U_α corresponds to a quasi-character of $\mathbb{Q}_p$). Obviously, the semi-simplification of $H_\ell(M(\chi)) \otimes \mathbb{C}$ is

$$J_\ell(M(\chi)) = \oplus_\alpha U_\alpha(\chi), \tag{15}$$

where $U_\alpha(\chi)$ denotes the twist of U_α by χ. Now

$$L_p(M(\chi), s) = \prod_\alpha L_p(U_\alpha(\chi), s), \tag{16}$$

where $L_p(U_\alpha(\chi), s) = (1 - \alpha\chi^{-1}(p)p^{-s})^{-1}$. Also, we have

$$\epsilon_p(M(\chi), \psi^{(\rho)}) = \prod_\alpha \epsilon_p(U_\alpha(\chi), \psi^{(\rho)}). \tag{17}$$

This is because I_p operates on $H_\ell(M(\chi))$ via a finite quotient, and the ϵ_p of complex representations of the Weil group are multiplicative with respect to short exact sequences.

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By analogy with (4), we now define

$$\mathcal{L}_p^{(\rho)}(M(\chi)) = \prod_\alpha \mathcal{L}_p^{(\rho)}(U_\alpha(\chi)), \tag{18}$$

where

(a) if $\mathrm{ord}_p(\alpha) \geq 0$, $\mathcal{L}_p^{(\rho)}(U_\alpha(\chi)) = 1$;

(b) if $\mathrm{ord}_p(\alpha) < 0$,

$$\mathcal{L}_p^{(\rho)}(U_\alpha(\chi)) = L_p(U_\alpha(\chi))/(\epsilon_p(U_\alpha(\chi), \psi^{(\rho)})L_p(\check{U}_\alpha(1)(\chi^{-1})).$$

Note that the $\mathcal{L}_p^{(\rho)}(U_\alpha(\chi))$ are always defined, i.e. in case (b), $L_p(U_\alpha(\chi))$ cannot have a pole because $\mathrm{ord}_p(\alpha) < 0$.

Define $h_p(M)$ to be the number of α's in $P(M)$, counted with multiplicity, such that $\mathrm{ord}_p(\alpha) < 0$. The next lemma relates this modified Euler factor to that in [1].

Lemma 3.

(i) $\mathcal{L}_p^{(\rho)}(M)/L_p(M) = \prod_{\substack{\alpha \\ \mathrm{ord}_p(\alpha) \geq 0}} (1 - \alpha) \times \prod_{\substack{\alpha \\ \mathrm{ord}_p(\alpha) < 0}} (1 - \frac{1}{p\alpha})$, ← correction factor ? (different from [G-S] & [G since they use arithm Frob send $\alpha \mapsto \frac{1}{\alpha}$)

(ii) *If χ is a non-trivial Dirichlet character whose conductor $C(\chi) = p^{a(\chi)}$ is a power of p, we have*

$$\mathcal{L}_p^{(\rho)}(M(\chi))/L_p(M(\chi)) = G_\rho(\chi^{-1})^{-h_p(M)} \times \Big(\prod_{\substack{\alpha \\ \mathrm{ord}_p(\alpha) < 0}} \alpha\Big)^{-a(\chi)},$$

where $G_\rho(\chi^{-1})$ is the Gauss sum

$$G_\rho(\chi^{-1}) = \sum_{x \bmod C(\chi)} \chi^{-1}(x)\exp(-2\pi\rho x/C(\chi)).$$

Proof. (i) is immediate from the definitions since $\epsilon_p(U_\alpha, \psi^{(\rho)}) = 1$ because U_α is an unramified quasi-character of $\mathbb{Q}_p$. By a standard formula (e.g. (3.4.6) on p. 15 of [4]),

$$\epsilon_p(U_\alpha(\chi), \psi^{(\rho)}) = \epsilon_p(\chi_p, \psi_p^{(\rho)}) \det U_\alpha(\mathrm{Frob}_p^{-a(\chi)}).$$

Also $\chi(p) = 0$ and a standard calculation shows that, since $C(\chi)$ is a power of p, $\epsilon_p(\chi_p, \psi_p^{(\rho)}) = G_\rho(\chi^{-1})$. Thus (ii) follows. ∎

We now define

$$\Lambda_{(p,\infty)}^{(\rho)}(M) = \mathcal{L}_\infty^{(\rho)}(M)\mathcal{L}_p^{(\rho)}(M)\Lambda(M)/(L_\infty(M)L_p(M)). = \frac{\mathcal{L}_p^{(\rho)}(M)}{L_p(M)} \mathcal{L}_\infty^{(\rho)}(M) L(M) \tag{19}$$

In view of our construction of the modified Euler factors, we clearly have the functional equation

$$\Lambda_{(p,\infty)}^{(\rho)}(M) = \prod_{v \neq p,\infty} \epsilon_v(M, \psi^{(\rho)}). \Lambda_{(p,\infty)}^{(-\rho)}(\check{M}(1)); \tag{20}$$

here we have assumed $ord_p(\alpha) \in \mathbb{Z}$ for all $\alpha \in P(M)$.

3. Conjecture about p-adic L-functions

Our aim is to express (and correct at the same time) the principal conjecture of [1] in terms of the function $\Lambda^{(\rho)}_{(p,\infty)}(M)$.

We assume now that p is *ordinary* for M, and begin by recalling what we mean by this (in many cases, much of our definition is redundant because of work of Bloch, Kato, Fontaine, Messing, Faltings, ...). Let ψ_p be the local cyclotomic character giving the action of $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the group μ_{p^∞} of all p-power roots of unity. Then p is good ordinary for M if the following conditions hold:

good → (i) I_p operates trivially on $H_\ell(M)$ for all $\ell \neq p$;

(ii) there exists a decreasing filtration $F^m H_p(M)$ $(m \in \mathbb{Z})$ of $H_p(M)$, which is stable under the action of $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and which is such that I_p operates on the m-th graded piece F^m/F^{m+1} by ψ_p^m;

(iii) for each $m \in \mathbb{Z}$, the $\mathbb{Q}_p$-dimension of F^m/F^{m+1} is equal to the complex Hodge number $h(-m, \omega(M)+m)$;

(iv) for each $m \in \mathbb{Z}$, the number of $\alpha \in P(M)$, counted with multiplicity, such that $\mathrm{ord}_p(\alpha) = m$ is equal to the Hodge number $h(m, \omega(M)-m)$.

Lemma 4.
(a) *The number of $\alpha \in P(M)$, counted with multiplicity, such that* $\mathrm{ord}_p(\alpha) < 0$ *is equal to* $d^+(M)$;

(b) *Let χ be a Dirichlet character and $n \in \mathbb{Z}$ be such that $\chi(-1) = (-1)^n$ and $M(n)(\chi)$ is critical at $s = 0$. Then $\alpha \in P(M)$ satisfies $\mathrm{ord}_p(\alpha) < 0$ if and only if $\mathrm{ord}_p(\alpha) < n$.*

Proof. (a) follows from (iv) and (11). (b) follows from (iv) and Lemma 2. ∎

Lemma 5. *Let χ be a character of p-power conductor and $n \in \mathbb{Z}$ be such that $\chi(-1) = (-1)^n$ and $M(n)(\chi)$ is critical at $s = 0$. Then*

$$\Lambda^{(\rho)}_{(p,\infty)}(M(n)(\chi))\Omega^{(\rho)}(M)^{-1} \tag{21}$$

does not depend on the choice of $\rho = \pm i$. L_p a priori

Proof. If $\chi = 1$, then n is even, and the lemma follows from (13). It $\chi \neq 1$, combine (13) with (ii) of Lemma 3, and note that $h_p(M) = d^+(M)$. ∎

Recall that $\mathbb{Q}(\mu_{p^\infty})$ is the maximal abelian extension of $\mathbb{Q}$, which is unramified outside p and ∞. Put

$$\mathcal{G} = G(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}). \tag{22}$$

We can view each Dirichlet character of p-power conductor as a p-adic valued character of $\mathcal{G}$, via the embedding (14). Let $\longrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$$\psi: \mathcal{G} \to \mathbb{Z}_p^* \tag{23}$$

be the cyclotomic character giving the action of $\mathcal{G}$ on μ_{p^∞}

Principal Conjecture. *If $\omega(M)$ is even, assume that $\mathbb{Q}(-\omega(M)/2)$ is not a direct summand of M. For each choice of the period $C^+(M)$, there exists a p-adic valued measure $\mu_{C^+(M)}$ on $\mathcal{G}$ such that*

$$\text{“}L_p(n,M,\chi)\text{”} = \int_{\mathcal{G}} \chi\psi^n d\mu_{C^+(M)} = \frac{\Lambda^{(\rho)}_{(p,\infty)}(M(n)(\chi))}{\Omega^{(\rho)}(M)} = L_\infty^{(\rho)}(M)\, L_p^{(\rho)}(M) \prod_{\ell \neq p,\infty} L_\ell(M) \quad (\rho = \pm i) \tag{24}$$

for all Dirichlet characters χ of p-power conductor and all $n \in \mathbb{Z}$ such that $\chi(-1) = (-1)^n$ and $M(n)(\chi)$ is critical at $s = 0$.

Note that Lemma 5 shows that the right hand side of (24) is independent of the choice of $\rho = \pm i$. This is essentially the principal conjecture of [1], expressed in our new normalization. However, the power of i given in the conjecture of [1] is not always correct, since it does not fully take into account the ε-factores at infinity.

Finally, we interpret the complex-functional equation p-adically. Having made choices of $C^+(M)$ and $C^+(\check{M}(1))$, there will then exist (assuming Conjecture 6.6 of [3] is valid) a non-zero rational number γ, independent of the choice of ρ, so that

$$\Omega^{(-\rho)}(\check{M}(1)) = \gamma\Omega^{(\rho)}(M)/(\prod_{v \neq \infty} \varepsilon_v(M, \psi^{(\rho)})). \tag{25}$$

Let $C(M)$ = the conductor of M, and let σ_M be the Artin symbol of $C(M)$ in $\mathcal{G}$.

Functional Equation. (p-adic version.) *Let χ be a Dirichlet character of p-power conductor, and let $n \in \mathbb{Z}$ be such that $\chi(-1) = (-1)^n$ and $M(n)(\chi)$ is critical at $s = 0$. Then, if $\phi = \chi\psi^n$, we have*

$$\text{“}L_p(n,M,\chi)\text{”} = \int_{\mathcal{G}} \phi \, d\mu_{C^+(M)} = \gamma^{-1}\phi^{-1}(\sigma_M) \underbrace{\int_{\mathcal{G}} \phi^{-1} d\mu_{C^+(\check{M}(1))}}_{\text{“}L_p(-n,\check{M}(1),\chi^{-1})\text{”}}.$$

Proof. Let $C(M) = \prod_v v^{a_v(M)}$. If $v \neq p, \infty$, then χ is unramified at p, and so

$$\varepsilon_v(M(n)(\chi), \psi^{(\rho)}) = \varepsilon_v(M, \psi^{(\rho)}) v^{-na_v(M)} \chi^{-1}(v^{a_v(M)}).$$

The p-adic functional equation now follows on applying (20) to $M(n)(\chi)$.

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